1.0

2.8

2.5

3.2

2.2

3.6

1.1

4.2

2.0

1.8

1.25

1.4

1.6

ASYMPTOTIC POWER OF EDF STATISTICS FOR EXPONENTIALITY

AGAINST WEIBULL AND GAMMA ALTERNATIVES

By

MICHAEL A. STEPHENS

TECHNICAL REPORT NO. 297

February 4, 1981

DEPARTMENT OF STATSITICS

STANFORD UNIVERSITY

STANFORD, CALIFORNIA

In recent papers, asymptotic distributions of EDF statistics, i.e. goodness-of-fit statistics based on the empirical distribution function, have been found by expanding a related function in a series of orthogonal functions. Suppose an ordered sample $y_1 \leq y_2 \leq \cdots \leq y_n$ is given, and we wish to test H_0: the parent population is the (continuous) distribution $F(y;\underset{\sim}{\theta})$, where $\underset{\sim}{\theta}$ is a vector of parameters. The transformation $x_i = F(y_i;\underset{\sim}{\theta})$, if $\underset{\sim}{\theta}$ is known, gives a set x_i of ordered uniform variates. Let $F_n(x)$ be the EDF of the x-values, and let $y_n(x) = \sqrt{n} \{F_n(x) - x\}$. The test statistics to be considered are W_n^2 , U_n^2 , and A_n^2 , of the Cramer-von Mises type, and can be defined in terms of $y_n(x)$:

$$W_n^2 = \int_0^1 \{y_n(x)\}^2 \, dx \ ,$$

$$U_n^2 = \int_0^1 \{y_n(x) - \bar{y}\}^2 \, dx \ ,$$

$$A_n^2 = \int_0^1 \{y_n(x)\}^2 \, w(x) \, dx \ ,$$

where

$$\bar{y} = \int_0^1 y_n(x) \, dx \qquad \text{and} \qquad w(x) = 1/(x-x^2) \ .$$

As $n \to \infty$, $y_n(x)$ approaches a Gaussian process $y(x)$, and the asymptotic distributions of the statistics are found from the integrals above with $y(x)$ replacing $y_n(x)$. This is done by first expanding the appropriate integrand as a series of orthogonal functions. The method was given by Watson (1967), and was developed in this connection by Durbin and Knott

-1-

(1972), by Stephens (1974), and most recently by Durbin, Knott and Taylor (1975). We shall illustrate the procedure with W_n^2 . Suppose $y_n(x)$ is expanded as a Fourier series:

$$y_n(x) = \sum_{j=1}^{\infty} b_j \sin \pi j x , \qquad 0 < x < 1 ;$$

then

$$b_j = \sqrt{2} \, z_{nj}/j\pi$$

where

(1)
$$z_{nj} = \sqrt{\frac{2}{n}} \sum_{i=1}^{n} \cos j\pi x_i .$$

Also

(2)
$$W_n^2 = \sum_{j=1}^{\infty} \frac{z_{nj}^2}{j^2 \pi^2} .$$

Durbin and Knott refer to z_{nj} as <u>components.</u> Stephens (1974) expanded $y_n(x)$ as a Fourier series with both sine and cosine terms to deal with U_n^2 ; for A_n^2, one expands $y_n(x)\{w(x)\}^{1/2}$ in associated Legendre functions. This approach to asymptotic theory of these statistics is a different approach from that taken by Anderson and Darling (1952). In the latter paper it was shown that if W^2 is a random variable with the asymptotic distribution of W_n^2 , it was possible to express W^2 as

(3)
$$W^2 = \sum_{j=1}^{\infty} \frac{z_j^2}{j^2 \pi^2}$$

where z_j are i.i.d. standard normal variables; the newer method gives a finite-n form of (3), and one can see the approach of the z_{nj} to the z_j .

In Durbin, Knott and Taylor (1975; henceforth referred to as DKT) the above technique is extended to the case where one or more components of $\underset{\sim}{\theta}$ are unknown and must be estimated from the data. Again the new approach offers an elegant and interesting alternative to the theory of unknown-parameter situations, begun by Darling (1955) and developed by Sukhatme (1972) and Stephens (1976). A modern treatment, particularly of the behavior of $y_n(x)$, is in Durbin (1973). DKT examined the distributions of W^2, U^2 and A^2 (absence of a subscript indicates asymptotic distributions) for tests for normality, and also gave some asymptotic power results for certain alternatives tending, as $n \to \infty$, to the null hypothesis. Similar theory was also given for tests of exponentiality, i.e. that y has distribution $F(y) = 1 - \exp(-\theta y)$, $y \geq 0$, but for W^2 and U^2 only. In this paper we examine A^2 for this situation, and compare asymptotic powers of W^2 , U^2 and A^2 against Weibull and Gamma alternatives. In repeating some of the DKT calculations, we have found that the coefficients given to calculate components, both for normal and exponential tests, are not correct; apparently the wrong matrix has been printed, in DKT Tables 1 and 6.

In order to save space, we follow as closely as possible the DKT notation, and simplify it for the special case of interest; the reader is referred to the DKT paper for theoretical details.

3

2. Components of W_n^2, when parameters must be estimated.

We continue by illustrating the general theory with W^2. Let unknown components of $\underset{\sim}{\theta}$ be estimated by maximum likelihood, and let $\hat{\underset{\sim}{\theta}}$ be the $\underset{\sim}{\theta}$ so calculated; write $\hat{z}_i = F(y;\hat{\underset{\sim}{\theta}})$, and suppose $\hat{Y}_n(x)$, $\hat{Y}(x)$, $\hat{W}_n^2$, $\hat{W}^2$ refer to the quantities defined in Section 1, but with $\hat{\theta}$ replacing θ. Then

$$(4) \qquad \hat{W}_n^2 = \int_0^1 \{\hat{Y}_n(x)\}^2 \, dx \quad \text{and} \quad \hat{W}^2 = \int \{\hat{Y}(x)\}^2 \, dx.$$

DKT show that

$$(5) \qquad \hat{W}_n^2 = \sum_{j=1}^{\infty} \mu_j \, \hat{z}_{nj}^2$$

and asymptotically

$$(6) \qquad \hat{W}^2 = \sum_{j=1}^{\infty} \mu_j \, \hat{z}_j^2$$

Where the μ_j and the components $\hat{z}_{nj}$, which are asymptotically $\hat{z}_j$, are to be found. The $\hat{z}_j$ are obtained first, as follows. Define $\lambda_j = (j\pi)^{-2}$ and $\ell_j(x) = \sqrt{2} \sin(\pi j x)$, $j=1,2,\ldots$; the λ_j are the weights of the z_{nj}^2 in representation (2). Define vectors ζ_n, and ζ with j-th components ζ_{nj} and ζ_j :

$$(7) \qquad \zeta_{nj} = \lambda_j^{-\frac{1}{2}} \int_0^1 \hat{Y}_n(x) \, \ell_j(x) \, dx,$$

4

and

(8)
$$\zeta_j = \lambda_j^{-\frac{1}{2}} \int_0^1 \hat{Y}(x) \, \ell_j(x) \, dx \ .$$

The $\hat{z}_j$ will be found as a linear combination of the ζ_j, and then $\hat{z}_{nj}$ will be defined as the same linear combination of the ζ_{nj}.

Suppose, for our purposes, that in the distribution $F(v;\theta)$, density $f(y:\theta)$, the vector θ contains only two components, so that $\theta' = (\theta_1,\theta_2)$; θ_1 is known, and on H_0 is θ_{10}, and θ_2 is to be estimated by maximum likelihood, with estimate $\hat{\theta}_2$. Let $\theta_0' = (\theta_{10},\theta_2)$ and $\hat{\theta}' = (\theta_{10},\hat{\theta}_2)$. Let F, f be $F(y;\theta)$ and $f(y;\theta)$ respectively; define

(9) $\qquad g_1(x) = \left(\dfrac{\partial F}{\partial \theta_1}\right)_{\theta=\theta_0} \qquad$ and $\qquad g_2(x) = \left(\dfrac{\partial F}{\partial \theta_2}\right)_{\theta=\theta_0}$,

with $x = F(y;\theta_0)$ making g_1 and g_2 functions of x; also let

(10) $\qquad I = -E\left(\dfrac{\partial^2 \ell n f}{\partial \theta_2^2}\right)_{\theta=\theta_0} \qquad ; \qquad I_{21} = E\left(\dfrac{\partial \ell n f}{\partial \theta_2} \cdot \dfrac{\partial \ell n f}{\partial \theta_1}\right)_{\theta=\theta_0} .$

Suppose vector Δ has components δ_j, for $j=1, 2, \ldots$

(11)
$$\delta_j = \lambda_j^{-\frac{1}{2}} \int_0^1 g_2(x) \, \ell_j(x) \, dx$$

It can be shown that $I = \Delta'\Delta = \sum_{j=1}^{\infty} \delta_j^2$. Then, from DKT, equation 5.1, adapted for only one unknown parameter,

(12)
$$E(\zeta\zeta') = T - \Delta\Delta'/I = M$$

5

where $\underset{\sim}{T}$ is the identity matrix of infinite order. Let $\underset{\sim}{\lambda}^r$ be the diagonal matrix with diagonal elements λ_1^r, λ_2^r, ... where $\lambda_1 \geq \lambda_2 \geq \ldots$, and let $\underset{\sim}{N}$ be the diagonal matrix which diagonalises $\underset{\sim}{\lambda}^{\frac{1}{2}} \underset{\sim}{M} \underset{\sim}{\lambda}^{\frac{1}{2}}$, i.e.

$$(13) \qquad \underset{\sim}{\lambda}^{\frac{1}{2}} \underset{\sim}{M} \underset{\sim}{\lambda}^{\frac{1}{2}} = \underset{\sim}{N} \underset{\sim}{\mu} \underset{\sim}{N}' .$$

where μ is a diagonal matrix with elements $\mu_1 \geq \mu_2 \geq \ldots$ on the diagonal. Finally, define

$$(14) \qquad \underset{\sim}{B} = \underset{\sim}{\mu}^{-\frac{1}{2}} \underset{\sim}{N}' \underset{\sim}{\lambda}^{\frac{1}{2}}$$

and

$$(15) \qquad \hat{\underset{\sim}{z}} = \underset{\sim}{B} \underset{\sim}{\zeta} \qquad ; \qquad \hat{\underset{\sim}{z}}_n = \underset{\sim}{B} \underset{\sim}{\zeta}_n .$$

These are the vectors whose components appear in (5) and (6).

Note that $E(zz') = B M B' = T$, using (12) for the first equality and (13) and (14) for the second. Recall that $\hat{x}_i = F(y_i ; \hat{\theta})$; DKT show that

$$\zeta_{nj} = \frac{\sqrt{2}}{n} \sum_{i=1}^{n} \cos \pi j \, \hat{x}_i$$

so that, using this and (15), $\hat{z}_n$ can be calculated from a sample (theoretically infinite) of y-values. DKT advocate using the components $\hat{z}_{nj}$ as test statistics for H_0; the above results show they are asymptotically $N(0,1)$ and independent.

6

Power calculations

For asymptotic power calculations we suppose the known parameter θ_1 in $\underset{\sim}{\theta}$ takes on values $\theta_1 = \theta_{10} + \gamma/\sqrt{n}$, where γ is a constant, as $n \to \infty$. Thus in the limit $\theta_1 = \theta_{10}$, the value on H_0. As $n \to \infty$, each component $\hat{z}_j$ will be independently distributed $N(\gamma \hat{\delta}_j, 1)$ where $\hat{\delta}_j$ is found below, so that asymptotic power can be found for individual components and for the entire statistic $\hat{W}^2$. To obtain $\hat{\delta}_j$ we define

$$\delta(x) = g_1(x) - I_{21}\, g_2(x)/I \quad.$$

The asymptotic process $\hat{y}(x)$ is Gaussian with mean $\gamma\, \hat{\delta}(x)$ and covariance

$$E\{\hat{y}(s),\ \hat{y}(t)\} = \min(s_1\ t) - st - g_2(s)\, g_2(t)/I \quad;$$

this follows from much more general results of Durbin (1973) quoted in DKT. Define $b_j = \int_0^1 \ell_j(x)\hat{\delta}(x)dx$, and let b_j be the j-th component of vector $\underset{\sim}{b}$; $\hat{\delta}_j$ will then be the j-th component of $\hat{\underset{\sim}{\delta}}$ given by (DKT, equation 6.4)

$$(16) \qquad\qquad \hat{\underset{\sim}{\delta}} = \underset{\sim}{\mu}^{-\frac{1}{2}} \underset{\sim}{N}'\ \underset{\sim}{b}$$

and, on H_1, $E(\hat{\underset{\sim}{z}}) = \gamma\ \hat{\underset{\sim}{\delta}}$. Let the length of $\hat{\underset{\sim}{\delta}}$,

$$\text{i.e. } \{\sum_{j=1}^{\infty} \hat{\delta}_j^2\}^{\frac{1}{2}}, \text{ be } |\hat{\underset{\sim}{\delta}}| \ .$$

3. Practical Calculations: Comments

(a) In the calculations above we work in principle with infinite matrices, beginning with $\underset{\sim}{M}$. Obviously in practice we take $\underset{\sim}{M}$ to be a square matrix of order q, say.

DKT suggest (bottom of P. 224) that this be obtained using $\underset{\sim}{\Delta}$ of length q, say $\underset{\sim}{\Delta}_q$, so that I is replaced by $I_q = \sum_{j=1}^{q} \varepsilon_j^2$, and $\underset{\sim}{T}$ by $\underset{\sim}{T}_q$, an identity matrix of order q. Let the resulting matrix $\underset{\sim}{T}_q - \underset{\sim}{\Delta}_q \underset{\sim}{\Delta}_q' / \underset{\sim}{I}_q$ be $\underset{\sim}{M}_q$; it may be used to continue the calculations, using the $q \times q$ top left hand corners of $\underset{\sim}{\lambda}$ and $\underset{\sim}{\mu}$, and finally gives a $q \times q$ matrix $\underset{\sim}{B}_q$ from (14). Let the first q components of $\underset{\sim}{\zeta}$ be a vector $\underset{\sim}{\zeta}_q$, then a q-vector $\underset{\sim}{\hat{z}}_q$ is given by $\underset{\sim}{\hat{z}}_q = \underset{\sim}{B}_q \underset{\sim}{\zeta}_q$ and in practice this vector gives q components, used, say, for testing purposes.

(b) We can however find I exactly from (10), and use it to give, say, $\underset{\sim}{M}_q^* = \underset{\sim}{T}_q - \underset{\sim}{\Delta}_q \underset{\sim}{\Delta}_q' / \underset{\sim}{I}$, and ultimately $\underset{\sim}{B}_q^*$ and $\underset{\sim}{\hat{z}}_q^*$ following the same steps as for $\underset{\sim}{B}_q$ and $\underset{\sim}{\hat{z}}_q$. It seems to us preferable to do this, since then $\underset{\sim}{M}_q^*$ is the correct asymptotic covariance matrix of the first q components of $\underset{\sim}{\zeta}$, whereas $\underset{\sim}{M}_q$ is only an approximation. Also the resulting $\underset{\sim}{\hat{z}}_q^*$ has asymptotic covariance matrix $\underset{\sim}{T}_q$ (the identity matrix), since $E(\underset{\sim}{z}_q^* \underset{\sim}{z}_q^{*\prime}) = \underset{\sim}{B}_q^* E(\underset{\sim}{\zeta}_q \underset{\sim}{\zeta}_q')\underset{\sim}{B}_q^{*\prime} = \underset{q}{B^*}\underset{q}{M^*}\underset{q}{B^{*\prime}} = \underset{\sim}{T}_q$; the asymptotic covariance matrix of $\underset{\sim}{\hat{z}}_q$ is slightly different from $\underset{\sim}{T}_q$.

(c) Ultimately we must extrapolate, as $q \to \infty$, to obtain values of μ_j which are used to obtain percentage points of $\hat{W}^2$; we used $q = 10, 12, 20,$ and 40 and the extrapolated μ_j were checked against eigenvalues obtained from the Darling approach; these are given in

8

Stephens (1976). DKT and Stephens, by different methods, use the μ_j and (6) to give percentage points for $\hat{W}^2$; the two sets match very well for all practical purposes.

(d) As stated earlier, DKT suggest the use of $\hat{z}_{nj}$ as test statistics; they give coefficients in their Tables 1 and 6, from which these components, for $i=1,2,\ldots,10$, are to be calculated. Unfortunately the values in the tables appear to be wrong. They should be the first rows of matrix $\underset{\sim}{B}$ (modified slightly to deal with the finiteness of $\underset{\sim}{\zeta}$ in practical calculations) and DKT discuss this on P. 228; however, they appear to have given the rows of matrix $\underset{\sim}{N}'$ and not those of $\underset{\sim}{B}$. Thus the components as given will not be independent nor have asymptotic variances 1, on H_0; and on the alternative, their means will not be $\gamma \delta_j$.

(e) Any orthogonal transformation, say $\underset{\sim}{H}$, applied to the $\hat{\underset{\sim}{z}}$ of (15) will give a new set of components $\underset{\sim}{p} = \underset{\sim\sim}{Hz}$ with asymptotic covariance matrix equal to the identity matrix, on both H_0 and H_1. If we choose the top row of $\underset{\sim}{H}$ to be the unit vector $\underset{\sim}{u}' = \hat{\underset{\sim}{\delta}}/|\hat{\underset{\sim}{\delta}}|$, then $E(p_1)$ becomes $\gamma \underset{\sim}{u}' \hat{\underset{\sim}{\delta}} = \gamma |\hat{\underset{\sim}{\delta}}|$; since $\underset{\sim}{u}$ is parallel to $\hat{\underset{\sim}{\delta}}$, this is the maximum mean attainable for given γ, by a single component. It can be used to calculate the maximum power attainable by a single component, for a given alternative; the alternative enters in the calculations of $\hat{\delta}(x)$ and hence of $\hat{\delta}_j$. We shall show later that this power is almost equal to that achieved by the Likelihood Ratio test, for Gamma and Wiebull alternatives. Thus the best single component is almost as effective as the L.R. test.

(f) Components have great mathematical interest, and are especially useful in power calculations, as we see later. We are not however, enthusastic for their practical use as test statistics. They are very tedious to calculate, and the calculations take one far away from the original data, so that the final values are difficult to interpret; these are factors which will not appeal to the practical user. Further, although they will have occasionally better power than the entire statistic, e.g. $\hat{W}^2$, for some alternatives, this is probably not true in general. Stephens (1974) illustrates this point in connection with tests where θ is completely specified.

We now give power calculations for both Wiebull and Gamma alternatives.

4.　**Power against the Weibull alternative.**

For the Weibull alternative we have

$$F(y;\underset{\sim}{\theta}) = 1-\exp\{-(y/\theta_2)^{\theta_1}\}$$

where

$$\theta_1 = 1+\gamma/\sqrt{n} \quad (\text{i.e.} \quad \theta_{10} = 1).$$

Then　(DKT, section 9)

$$g_1(x) = -(1-x)\ell n(1-x)\ell n\{-\ell n(1-x)\} \; ; \; g_2(x) = (1-x)\ell n(1-x);$$

and it may be shown that

$$I = 1, \quad I_{21} = \tilde{\gamma}-1$$

where $\tilde{\gamma}$ is Euler's constant 0.577216. (DKT, Equation (9.4), gives
I incorrectly). These have been used to give values of $\underset{\sim}{\hat{\delta}}$ for
$q = 10,12,20,40$ (as before, q is the order of magnitude in the
matrix calculations) and these were extrapolated to give $\underset{\sim}{\hat{\delta}}$ for
$q \to \infty$. The first four components of $\underset{\sim}{\hat{\delta}}$ are given in Table 1. For
individual components, the asymptotic power can be calculated since
z_j has a $N(\gamma\hat{\delta}_j,1)$ distribution.　DKT　choose γ to be the values
required to give power equal to 0.5 and 0.95 for the appropriate
L.R. test of θ_1. This test is based on the M.L. estimator $\hat{\theta}_1$ of
θ_1 for which, asymptotically

$$\sqrt{n}\ (\hat{\theta}_1 - 1) \quad \text{is distributed} \quad N(0,\ 0.608)\ .$$

(Johnson and Kotz, 1970, Vol. 1, p. 254). Then for 0.5 power $\gamma = 1.283$ and for 0.95 power $\gamma = 2.563$ for one-sided tests; for two sided tests $\gamma = 1.528$ and 2.811. These are the values corresponding to the results in the columns of DKT, Table 8. We have repeated some of these calculations and the results are included in Table 2.

For the complete statistic $\hat{W}^2$ we have the asymptotic distribution represented by

$$\hat{W}^2 = \sum_{j=1}^{\infty} \mu_j \hat{z}_j^2$$

where the

$$\hat{z}_j = N(\gamma \hat{\delta}_j, 1)\ .$$

The asymptotic distribution can be calculated by a method due to Imhof, adapted for the case of an infinite set of components by Durbin and Knott (1972). Alternatively, we can approximate it by finding the cumulants of $\hat{W}^2$ and fitting a curve of the form $a + bX_p^2$, where a, b, p are chosen so that the first three cumulants match those of $\hat{W}^2$. This was done, e.g. in Stephens (1974), for power studies where all parameters were known, and a graph was given for power against γ. Note that cumulants K_j of $\hat{W}^2$ on H_1 are

$$K_1 = E(\hat{W}^2) = \sum \mu_j + \gamma^2 \sum \mu_j \hat{\delta}_j^2 = \sum \mu_j + \gamma^2 \sum c_j^2$$

12

where $c = \{c_1, c_2, \ldots\}' = N'b$ in (16), and all sums are for $j = 1, 2, \ldots, \infty$; also,

$$K_2 = 2 \sum \mu_j^2 + 4\gamma^2 \sum c_j^2 \mu_j,$$

and

$$K_3 = 8 \sum \mu_j^3 + 24\gamma^2 \sum c_j^2 \mu_j^2$$

so that for the cumulants it is not strictly necessary to know the $\hat{\delta}_j$, but only the c_j and μ_j.

We have done power calculations by both methods, and the results agreed very closely; a graph of power against γ^2 is given in Figure 1.

Calculations for U^2 and for A^2.

The calculations of μ_j and $\hat{\delta}_j$ for U^2 are based on the asymptotic behaviour of $\hat{y}_n(x) - \bar{y}$ where $\bar{y} = \int_0^1 \hat{y}_n(x) dx$, and those for A^2 on $\hat{y}_n(x) / \{x(1-x)\}^{\frac{1}{2}}$. The appropriate changes in the theory are given in DKT, Section 4 and we will not repeat them here. The statistic is still expressed in the form (6) with $\hat{W}^2$ replaced by $\hat{U}^2$ or $\hat{A}^2$, and asymptotically the components $\hat{z}_j$ have $N(\gamma \hat{\delta}_j, 1)$ distributions; the values of $\hat{\delta}_j$ are given in Table 1. We have also used the μ_j to give null-distribution points by Imhof's method as adapted by Durbin and Knott; the upper tail points at level α are then the following:

α (%):	10	5	2.5	1
% point:	1.061	1.321	1.590	1.947

13

5. Power against the Gamma alternative.

For the Gamma distribution, we have

$$(17) \qquad F(y;\theta) = \frac{1}{\Gamma(m)} \frac{1}{(\alpha)^m} \int_0^y e^{-z/\alpha} z^{m-1} dz$$

where, to simplify notation, we have let $\theta_1 = m$ and $\theta_2 = \alpha$. On H_0, $m=1$, and on H_1, $m=1+\gamma/\sqrt{n}$. We then have, with $x = F(y;\theta)$

$$g_1(x) = \frac{1}{\Gamma(m)} \int_0^y e^{-z} z^{m-1} \ln z \, dz - \frac{\Gamma'(m)}{\{\Gamma(m)\}^2} \int_0^y e^{-z} z^{m-1} dz$$

and $g_2(x) = (1-x)\ln(1-x)$ as before.

I becomes m/θ^2, which, for $m = 1$, $\theta = 1$ is 1;

$$I_{21} = \int_0^\infty y e^{-y} \ln y \, dy - \int_0^\infty e^{-y} \ln y \, dy$$

$$= \Gamma'(2) - \Gamma'(1) = (1-\gamma) - (-\gamma) = 1$$

Thus $\hat{\delta}(x) = g_1(x) - g_2(x)$ and for $\hat{\delta}_j$ we need integrals

$$J_1 = \int_0^1 g_1(x) \ell_j(x) dx \quad \text{and} \quad J_2 = \int_0^1 g_2(x) \ell_j(x) dx .$$

The second integral is already known from the calculations needed in Section 4; the first gives $J_1 = J_{11} - J_{12}$ where

$$J_{11} = \frac{1}{\Gamma(m)} \int_0^1 \int_0^y e^{-z} z^{m-1} \ln z \, dz \, \ell_j(x) dx$$

with $y = -\ln(1-x)$. The integral is calculated by reversing the order

14

of integration. Let $s = 1-e^{-z}$; when $m = 1$ we have

$$J_{11} = -\int_0^1 \ell n(-\ell n(1-s)) \frac{\sqrt{2}}{\pi j} (\cos \pi j - \cos \pi js) ds$$

The original integral has been reduced to a single integral, and although some limit problems remain at the end points, ft can be evaluated numerically. The integral J_{12} is

$$J_{12} = \Gamma'(1) \int_0^1 \int_0^y e^{-z} dz \, \ell_j(x) dx$$

$$= \Gamma'(1) \int_0^1 x \, \ell_j(x) dx$$

$$= \Gamma'(1) \sqrt{2} \, C/\pi j$$

where $C = -1$ if j is even and 1 if j is odd. Using these results we have found the $\hat{\delta}_j$ for the Gamma alternative, and thus the asymptotic power of components and the entire $\hat{W}^2$ can be evaluated.

For U^2 and A^2 the integrals are evaluated on the same lines as those above, though for A^2, where $\ell_j(x)$ consists of Ferrar associated Legendre functions, the algebra is more cumbersome. These details will be omitted. For comparison purposes we need the best test against the Γ-alternative. This is based on the M.L. estimate $\hat{m}$ of m, which asymptotically for $m = 1$ has distribution $N(1, \sigma^2)$ where $1/\sigma^2 = n\{\psi'(1)-1\} = 0.645n$; $\Psi'(x)$ is the digamma function (Johnson and Kotz, 1970, vol. 1, p. 188). Thus for the best one-sided 5% test, we reject if $\hat{m}-1 > 1.645\sigma$; on the

15

alternative the power is

$$Pr(\hat{m}-1 > 1.645\sigma) = P(\hat{m}-(1+\gamma/\sqrt{n}) > 1.645\sigma - \gamma/\sqrt{n})$$

$$= P\{(\hat{m}-(1+\gamma/\sqrt{n})/\sigma) > 1.645 - \gamma/(\sigma\sqrt{n})\} .$$

The left hand side of the inequality is a $N(0,1)$ variable;
thus for 50% power we have $\gamma = 1.645\sigma\sqrt{n} = 1.645/(0.645)^{1/2} = 2.060$;
for 95% power $1.645-\gamma/(\sigma\sqrt{n}) = -1.65$, giving $\gamma = 4.119$. For two-
tailed tests, $\gamma = 2.441$ and 4.489 for 50% and 95% power
respectively. Calculations on the lines of those for the Wiebull
alternative give power graphs against γ^2, shown in Figure 2.

Acknowledgments

References

Anderson, T. W. and Darling, D. A. (1952). Asymptotic theory of certain goodness of fit criteria based on stochastic processes. _Ann. Math. Statist._, 23, 193-212.

Darling, D. A. (1955). The Cramér-Smirnov test in the parametric case. _Ann. Math. Statist._, 26, 1-20.

Durbin, J. (1973). Weak convergence of the sample distribution function when parameters are estimated. _Ann. Statist._, 1, 279-290.

Durbin, J. and Knott, M. (1972). Components of Cramér-von Mises statistics, I. _J. R. Statist. Soc. B_, 34, 290-307.

Durbin, J., Knott, M., and Taylor, C. C. (1975). Components of Cramér-von Mises statistics, II. _J. R. Statist. Soc. B_, 37, 216-237.

Johnson, N. L. and Kotz, S. (1970). _Distributions in Statistics, Volume 1: Continuous Univariate Distributions_. Boston: Houghton Mifflin.

Stephens, M. A. (1974). Components of goodness-of-fit statistics. _Annals de l' Institut Henri Poincaré, Ser. B_, 10, 37-54.

Stephens, M. A. (1976). Asymptotic results for goodness-of-fit statistics with unknown parameters. _Ann. Statist._, 4, 357-369.

Sukhatme, S. (1972). Fredholm determinant of a positive definite kernel of a special type and its application. _Ann. Math. Statist._, 43, 1914-1926.

Watson, G. S. (1967). Some problems in the statistics of directions. _Bulletin of the Int. Statist. Inst. Conference_, 36th Session, Sydney, Australia.

Table 1

Values of $\hat{\delta}_j$ and of $|\hat{\underline{\delta}}|$, for the test of exponentiality vs. Weibull and Gamma alternatives

Statistic	j:	1	2	3	4	$|\hat{\underline{\delta}}|$
$\hat{W}^2$ (Weibull)	:	1.1322	0.1062	0.4176	0.0693	1.2798
$\hat{U}^2$ (Weibull)	:	0.9632	0.0645	0.5075	0.0555	1.2591
$\hat{A}^2$ (Weibull)	:	1.2050	0.0087	0.3479	0.0018	1.2823
$\hat{W}^2$ (Gamma)	:	0.6466	0.1917	0.2608	0.1237	0.7953
$\hat{U}^2$ (Gamma)	:	0.5836	0.0901	0.3211	0.0442	0.7866
$\hat{A}^2$ (Gamma)	:	0.7256	0.1610	0.2238	0.0801	0.7978

18

Table 2

Asymptotic powers of components, and of the entire statistics

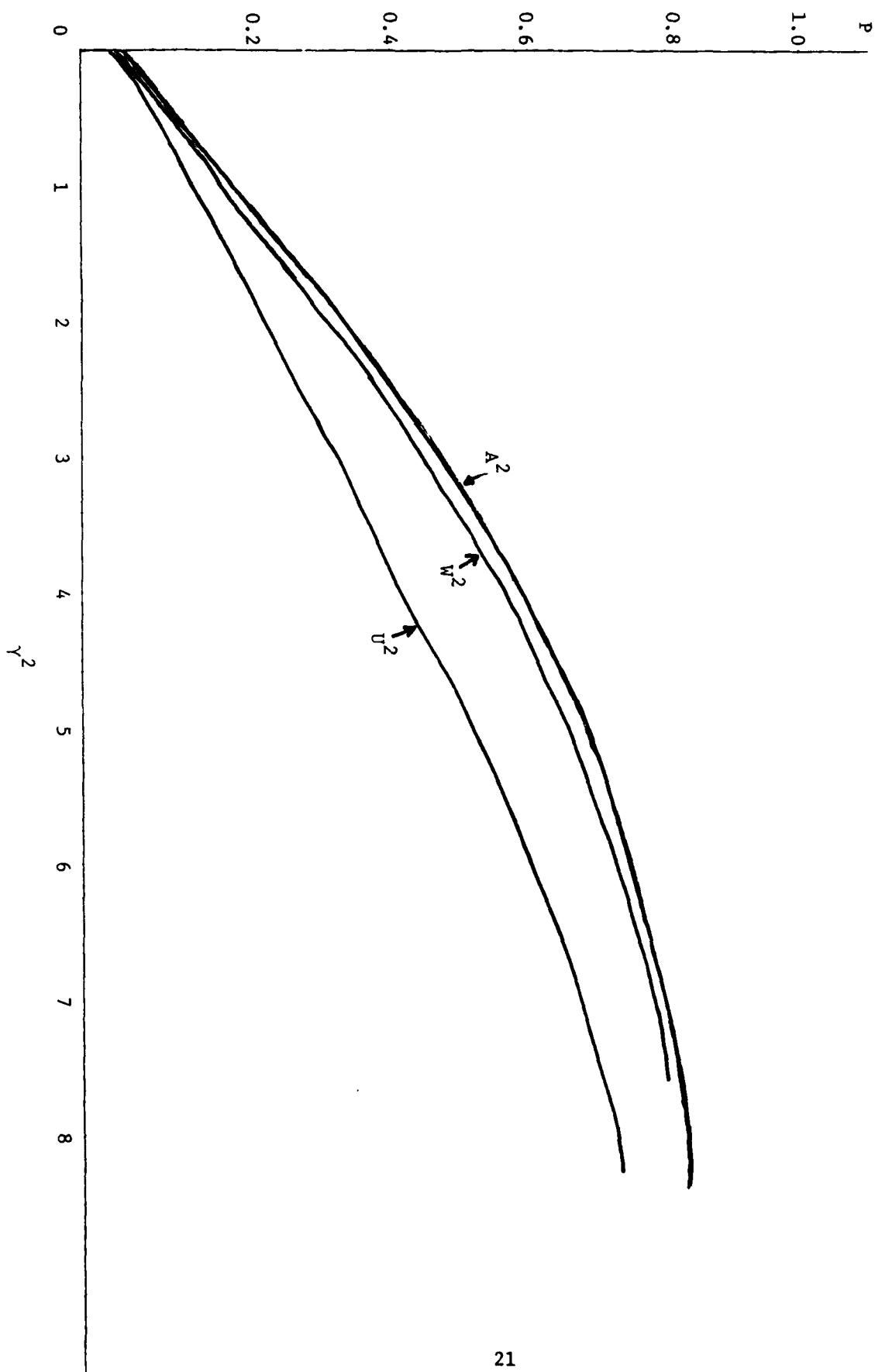$\hat{W}^2$, $\hat{U}^2$, and $\hat{A}^2$ in tests for exponentiality against Weibull alternatives

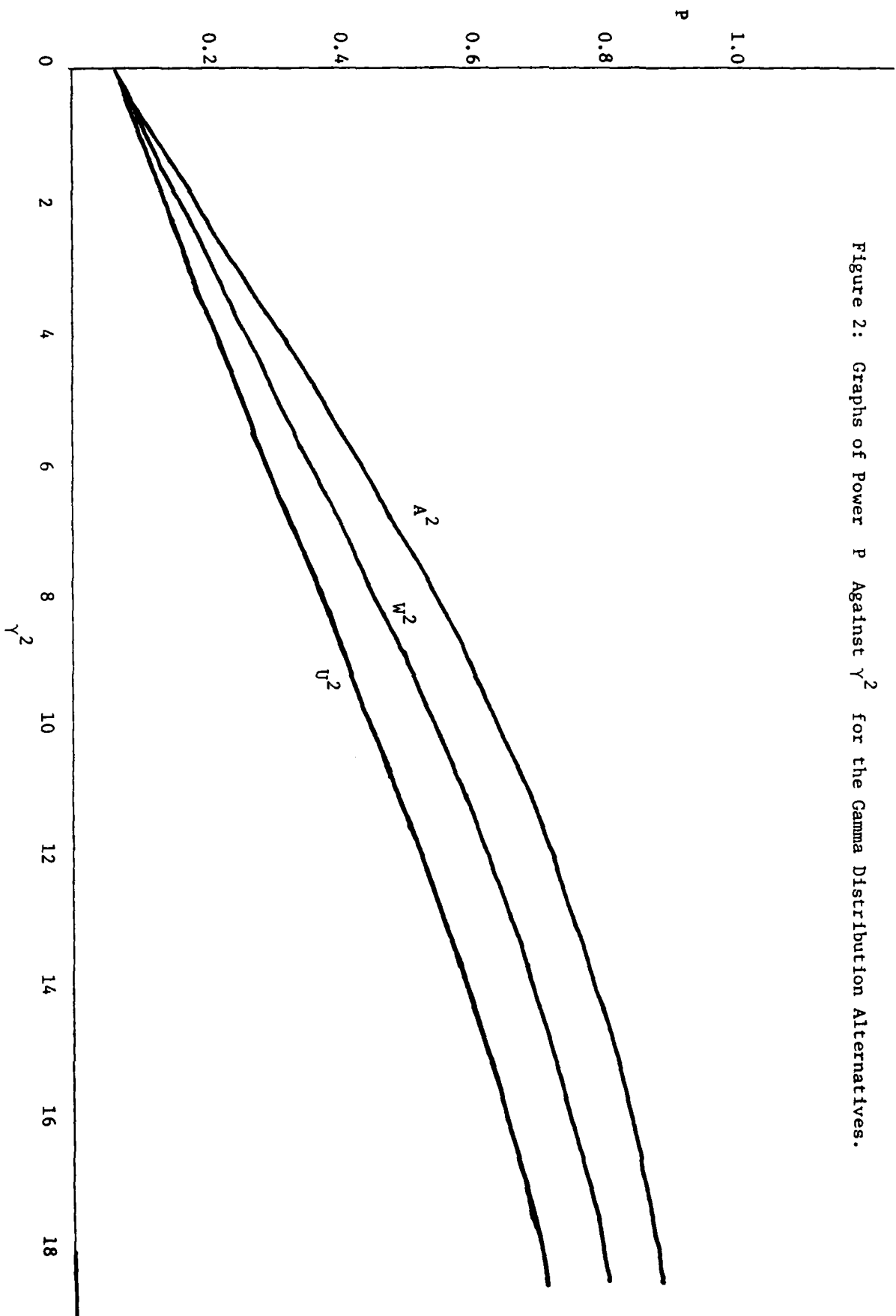|		One-Sided Test		Two-sided Test	
γ^2 :		1.645	6.580	2.335	7.901
Best test power:		.50	.95	.50	.95
$\hat{W}^2$	$\hat{z}_1$:	.424	.896	.409	.889
components	$\hat{z}_2$:	.066	.085	.053	.060
	$\hat{z}_3$:	.134	.283	.098	.217
	$\hat{z}_4$:	.060	.071	.051	.054
best:	$|\hat{\delta}|$	.499	.949	.498	.949
$\hat{U}^2$	$\hat{z}_1$:	.341	.796	.313	.773
components	$\hat{z}_2$:	.059	.070	.051	.054
	$\hat{z}_3$:	.160	.366	.121	.297
	$\hat{z}_4$:	.058	.067	.051	.053
best:	$|\hat{\delta}|$	.488	.944	.486	.943
	$\hat{z}_1$:	.460	.926	.453	.923
$\hat{A}^2$	$\hat{z}_2$:	.051	.052	.050	.050
components	$\hat{z}_3$:	.115	.226	.083	.165
	$\hat{z}_4$:	.050	.051	.050	.050
best:	$|\hat{\delta}|$	.500	.950	.500	.950
Entire	$\hat{W}^2$:	.296	.829	0.403	0.890
Statistics	$\hat{U}^2$	.216	.703	0.294	0.744
	$\hat{A}^2$	.307	.835	0.438	0.917

Table 3

Asymptotic powers of components, and of the entire statistics

$\hat{W}^2$, $\hat{U}^2$ and $\hat{A}^2$ in tests for exponentiality against Gamma alternatives.

		One-Sided Test		Two-Sided Test	
	γ^2 :	4.2420	16.9678	5.9565	20.1511
Best test power:		.50	.95	.50	.95
$\hat{W}^2$	$\hat{z}_1$	.377	.846	.352	.827
	$\hat{z}_2$	.106	.196	.075	.138
components	$\hat{z}_3$	.134	.284	.098	.216
	$\hat{z}_4$	.082	.128	.061	.086
best	$|\hat{\delta}|$	.497	.949	.493	.946
$\hat{U}^2$	$\hat{z}_1$	.329	.776	.297	.745
	$\hat{z}_2$	.072	.101	.056	.069
components	$\hat{z}_3$	.163	.374	.123	.302
	$\hat{z}_4$	.060	.072	.051	.055
best	$|\hat{\delta}|$	.490	.945	.484	.942
$\hat{A}^2$	$\hat{z}_1$	.440	.911	.425	.903
	$\hat{z}_2$	.095	.163	.068	.112
components	$\hat{z}_3$	.118	.235	.084	.171
	$\hat{z}_4$	.069	.094	.054	.065
best	$|\hat{\delta}|$	.499	.950	.495	.948
Entire	$\hat{W}^2$	.256	.772	.340	.908
Statistics	$\hat{U}^2$	.202	.674	.274	.841
	$\hat{A}^2$	.311	.853	.420	.754

Figure 1: Graphs of Power P Against γ for the Weibull Distribution Alternative.

A^2

W^2

U^2

P

γ^2

21

Figure 2: Graphs of Power P Against γ^2 for the Gamma Distribution Alternatives.

(14) TR-317

REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER 297	2. GOVT ACCESSION NO. AD-A097107	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) ASYMPTOTIC POWER OF EDF STATISTICS FOR EXPONENTIALITY AGAINST WEIBULL AND GAMMA ALTERNATIVES.		5. TYPE OF REPORT & PERIOD COVERED TECHNICAL REPORT
		6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) MICHAEL A. STEPHENS		8. CONTRACT OR GRANT NUMBER(s) N00014-76-C-0475
9. PERFORMING ORGANIZATION NAME AND ADDRESS Department of Statistics Stanford University Stanford, CA 94305		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS NR-042-267
11. CONTROLLING OFFICE NAME AND ADDRESS OFFICE Of Naval Research Statistics & Probability Program Code 436 Arlington, VA 22217		12. REPORT DATE FEBRUARY 4, 1981
		13. NUMBER OF PAGES 22
14. MONITORING AGENCY NAME & ADDRESS(if different from Controlling Office) (12) 26		15. SECURITY CLASS. (of this report) UNCLASSIFIED
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

APPROVED FOR PUBLIC RELEASE: DISTRIBUTION UNLIMITED.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

EDF Statistics, Goodness-of-fit, Power of tests for exponentiality.

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

PLEASE SEE REVERSE SIDE.

332580

297

ASYMPTOTIC POWER OF EDF STATISTICS FOR EXPONENTIALITY
AGAINST WEIBULL AND GAMMA ALTERNATIVES

Asymptotic distributions of EDF statistics — goodness-of-fit

statistics based on the empirical distribution function — can be

obtained by expanding a related function in a series of orthogonal

functions. This method was given by Watson and developed especially

in this connection by Durbin and Knott and by the author. An earlier

approach by Anderson and Darling is different and the newer methods

provide a sharper view of the approach to infinity. Similar approaches

were employed by Durbin, Knott and Taylor to test for normality and

also gave asymptotic power results for certain alternatives and in

some cases for tests of exponentiality. In this paper, asymptotic

powers of some EDF statistics for testing exponentiality against

Weibull and Gamma alternatives are developed.